Finite Frequency H_∞ Control for Vehicle Active Suspension Systems

Weichao Sun, Huijun Gao, *Senior Member, IEEE*, and Okyay Kaynak, *Fellow, IEEE*

***Abstract*—This brief addresses the problem of H_∞ control for active vehicle suspension systems in finite frequency domain. The H_∞ performance is used to measure ride comfort so that more general road disturbances can be considered. By using the generalized Kalman–Yakubovich–Popov (KYP) lemma, the H_∞ norm from the disturbance to the controlled output is decreased in specific frequency band to improve the ride comfort. Compared with the entire frequency approach, the finite frequency approach suppresses the vibration more effectively for the concerned frequency range. In addition, the time-domain constraints, which represent performance requirements for vehicle suspensions, are guaranteed in the controller design. A state feedback controller is designed in the framework of linear matrix inequality (LMI) optimization. A quarter-car model with active suspension system is considered in this brief and a numerical example is employed to illustrate the effectiveness of the proposed approach.**

***Index Terms*—Active suspension systems, constraints, finite frequency, generalized KYP lemma, H_∞ control.**

I. Introduction

A VEHICLE suspension system basically consists of wishbone, spring, and shock absorber to transmit and filter all forces between body and road. The spring is to carry the body-mass and to isolate the body from road disturbances and thus contributes to ride comfort. The task of the damper is the damping of body and wheel oscillations, where the avoidance of wheel oscillations directly refers to ride safety. Since the vehicle suspension system is responsible for ride comfort and safety, it plays an important role in modern vehicles.

In recent years, a lot of efforts have been made to develop models for suspension systems and to define design specifications that reflect the main objectives to be taken into account. In this sense, ride comfort, road-holding ability, suspension deflection, and actuator saturation are important factors to be addressed by any control scheme. However, these requirements are conflicting. For example, increasing ride comfort results in larger suspension stroke and smaller damping in the wheel-hop mode. Therefore, vehicle suspension design requires a compromise between ride comfort and vehicle control. To achieve a compromise between the performance requirements, a considerable amount of research has been carried out for the last few decades [3], [17], [21]. Among the proposed solutions, active suspension is a possible way to improve suspension performance and has attracted much attention [10], [19], [24], and many active suspension control approaches are proposed, based on various control techniques such as linear quadratic Gaussian (LQG) control [4], adaptive control and nonlinear control [11], fuzzy logic and neural network control [15], and H_∞ control [14]. In particular, H_∞ active suspensions have been intensively discussed in the context of robustness and disturbance attenuation [6], [7]. Therefore, in recent years, more and more attention has been devoted to the H_∞ control of active suspensions, and a number of important results have been reported, see for example, [5], [13] and the references therein.

The most important objective for vehicle suspension systems is the improvement of ride comfort. In other words, the main task is to design the controller which can succeed in stabilizing the vertical motion of the car body and isolating the force transmitted to the passengers as well. In the literature it is possible to find many results which aim at improving ride comfort [8], [20], [22]. These results can effectively achieve desired vehicle suspension performance, especially the ride comfort. It is worth mentioning that most of the reported approaches are considered in the entire frequency domain. However, active suspension systems may just belong to certain frequency band, and ride comfort is known to be frequency sensitive. From the ISO2361, the human body is much sensitive to vibrations of 4–8 Hz in the vertical direction. Hence, the development of H_∞ control in finite frequency domain is significative for active suspension systems.

The current approach for finite frequency domain is to introduce the weighting functions. The weighting method is useful in practice, however, the additional weights increase the system complexity. Besides, the process of selecting appropriate weights is time-consuming, especially when the designer has to shoot for a good tradeoff between the complexity of the weights and the accuracy in capturing desired specifications. An alternative approach is to grid the frequency axis. This approach has a practical significance especially when the system is well damped and the frequency response is expected to be smooth. But it lacks a rigorous performance guarantee in the design process.

Another approach that avoids both weighting functions and frequency gridding is to generalize the fundamental machinery, the Kalman–Yakubovič–Popov (KYP) lemma. The KYP lemma establishes the equivalence between a frequency domain inequality for a transfer function and a linear matrix inequality (LMI) associated with its state-space realization [1], [9], [12]. It allows us to characterize various properties of dynamic systems in the frequency domain in terms of LMIs. However, the standard KYP lemma is only applicable for the infinite frequency range. Recently, a very significant devel-

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W. Sun and H. Gao are with the Space Control and Inertial Technology Research Center, Harbin Institute of Technology, Harbin 150001, China (e-mail: 1984sunweichao@gmail.com; hjgao@hit.edu.cn).

O. Kaynak is with the Electrical and Electronic Engineering Department, Bogazici University, Bebek 80815, Istanbul, Turkey (e-mail: okyay.kaynak@boun.edu.tr).

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opment made by Iwasaki and Hara is the generalized KYP lemma [23]. It establishes the equivalence between a frequency domain property and an LMI over a finite frequency range, allowing designers to impose performance requirements over chosen finite or infinite frequency ranges. The generalized KYP lemma is very useful for the analysis and synthesis problems in practical applications.

Different from the conventional methodologies that consider the H_∞ control over the entire frequency range, in this brief, we consider the active suspension systems over the finite frequency range based on the generalized KYP lemma. In addition, the time-domain constraints (road holding, suspension deflection, and actuator saturation) are guaranteed in the controller design. By using the generalized KYP lemma, the frequency domain inequalities are transformed into linear matrix inequalities, and our attention is focused on developing methods to design a state feedback control law based on matrix inequalities such that the resulting closed-loop system is asymptotically stable with a prescribed level of disturbance attenuation in certain frequency domain. The effectiveness of the proposed approach is shown by a design example.

The remainder of this brief is organized as follows. The problem of finite frequency H_∞ controller design for active suspension systems is formulated in Section II. Section III presents controller design results. A design example illustrating the usefulness and advantage of the proposed methodology is given in Section IV and conclusions are given in Section V.

A. Notation

For a matrix P, P^T, P^{-1}, and $P^\perp$ denote its transpose, inverse and orthogonal complement, respectively; the notation $P > 0$ (≥ 0) means that P is real symmetric and positive definite (semi-definite); and $[P]_s$ means $P + P^T$. For a vector or matrix, $\{\cdot\}_i$ $(i = 1, 2, \ldots)$ represents the ith line of the vector or matrix, and $\|G\|_\infty$ denotes the H_∞-norm of transfer function matrix $G(s)$. In symmetric block matrices or complex matrix expressions, we use an asterisk $(*)$ to represent a term that is induced by symmetry and $\mathrm{diag}\{\ldots\}$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. The space of square-integrable vector functions over $[0, \infty)$ is denoted by $L_2[0, \infty)$, and for $w = \{w(t)\} \in L_2[0, \infty)$, its norm is given by $\|w\|_2 = \sqrt{\int_{t=0}^{\infty} |w(t)|^2 dt}$.

II. Problem Formulation

The quarter car model shown in Fig. 1 is considered in this brief. In Fig. 1, m_s is the sprung mass, which represents the car chassis; m_u is the unsprung mass, which represents mass of the wheel assembly; c_s and k_s are damping and stiffness of the suspension system, respectively; k_t and c_t stand for compressibility and damping of the pneumatic tyre, respectively; z_s and z_u are the displacements of the sprung and unsprung masses, respectively; z_r is the road displacement input; u is the active input of the suspension system. This model has been used extensively in the literature and captures many important characteristics of more detailed models. In this brief, the effect of actuator

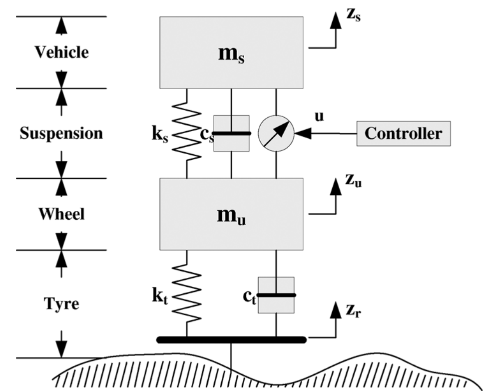

Fig. 1. Quarter-car model with an active suspension.

dynamics is neglected and the actuator is modelled as an ideal force generator.

Define the following state variables:

$$x_1(t) = z_s(t) - z_u(t),\ x_2(t) = z_u(t) - z_r(t)$$
$$x_3(t) = \dot{z}_s(t),\ x_4(t) = \dot{z}_u(t)$$

where $x_1(t)$ denotes the suspension deflection, $x_2(t)$ is the tire deflection, $x_3(t)$ is the sprung mass speed, and $x_4(t)$ denotes the unsprung mass speed. We define the disturbance inputs as $w(t) = \dot{z}_r(t)$. Then, by defining $x(t) = [x_1(t)\ \ x_2(t)\ \ x_3(t)\ \ x_4(t)]^T$, and according to the dynamic characteristic of the active suspension system, the state-space form can be given

$$\dot{x}(t) = Ax(t) + B_1 w(t) + Bu(t) \tag{1}$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -\frac{k_s}{m_s} & 0 & -\frac{c_s}{m_s} & \frac{c_s}{m_s} \\ \frac{k_s}{m_u} & -\frac{k_u}{m_u} & \frac{c_s}{m_u} & -\frac{c_s+c_t}{m_u} \end{bmatrix}$$
$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_s} \\ -\frac{1}{m_u} \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ \frac{c_t}{m_u} \end{bmatrix}. \tag{2}$$

It is widely accepted that ride comfort is closely related to the body acceleration in frequency band 4–8 Hz. Consequently, in order to improve ride comfort it is important to keep the transfer function from the disturbance inputs $w(t)$ to car body acceleration $\ddot{z}_s(t)$ as small as possible over the frequency band 4–8 Hz.

In order to make sure the car safety, we should ensure the firm uninterrupted contact of wheels to road, and the dynamic tire load should be small, that is $k_t(z_u(t) - z_r(t)) < (m_s + m_u)g$.

In addition, the structural features of the vehicle also constrain the amount of suspension deflection, that is

$|z_s(t) - z_u(t)| \leq z_{\max}$, where $z_{\max}$ is the maximum suspension deflection.

Another hard constraint imposed on active suspensions is from the limited power of the actuator, that is $|u(t)| \leq u_{\max}$.

In order to satisfy the performance requirements, the controlled outputs are defined by

$$z_1(t) = \ddot{z}_s(t),\ z_2(t) = \begin{bmatrix} \frac{z_s(t)-z_u(t)}{z_{\max}} & \frac{k_t(z_u(t)-z_r(t))}{(m_s+m_u)g} \end{bmatrix}^T. \quad (3)$$

Therefore, the vehicle suspension control system can be described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_1 w(t) \\ z_1(t) &= C_1 x(t) + D_1 u(t) \\ z_2(t) &= C_2 x(t) \end{aligned} \quad (4)$$

where A, B_1, and B are defined in (2), and

$$\begin{aligned} C_1 &= [\,-\tfrac{k_s}{m_s} \quad 0 \quad -\tfrac{c_s}{m_s} \quad \tfrac{c_s}{m_s}\,] \\ C_2 &= \begin{bmatrix} \frac{1}{z_{\max}} & 0 & 0 & 0 \\ 0 & \frac{k_t}{(m_s+m_u)g} & 0 & 0 \end{bmatrix} \\ D_1 &= \tfrac{1}{m_s}. \end{aligned}$$

Denote $G(j\omega)$ as the transfer function from the disturbance inputs $w(t)$ to the controlled output $z_1(t)$. The finite frequency H_∞ control problem is to design a controller such that the closed-loop system guarantees

$$\sup_{\varpi_1<\omega<\varpi_2} \|G(j\omega)\|_\infty < \gamma \quad (5)$$

where $\gamma > 0$ is a prescribed scalar, and ϖ_1, ϖ_2 represent the upper and lower bounds of the concerned frequency. In addition, from the safety and mechanical structure point of view, the constraints

$$|u(t)| \leq u_{\max},\ |\{z_2(t)\}_i| \leq 1,\ i = 1, 2 \quad (6)$$

need to be guaranteed.

III. Controller Design

To facilitate the presentation, we introduce the essential lemmas. For the sake of brevity, all the proof of the lemmas have been omitted.

Lemma 1: (Generalized KYP Lemma [23]) Consider the linear system ($\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$). Given a symmetric matrix Π, the following statements are equivalent.

1) The finite frequency inequality

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^T \Pi \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0,\ \varpi_1 \leq \omega \leq \varpi_2. \quad (7)$$

2) There exist symmetric matrices P and Q satisfying $Q > 0$ and

$$\begin{bmatrix} \Gamma[P, Q, \bar{C}, \bar{D}] & [\bar{C} \quad \bar{D}]^T \\ * & -I \end{bmatrix} < 0 \quad (8)$$

where

$$\Gamma[P, Q, \bar{C}, \bar{D}] = \begin{bmatrix} \bar{A} & \bar{B} \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -Q & P + j\varpi_c Q \\ P - j\varpi_c Q & -\varpi_1\varpi_2 Q \end{bmatrix} \cdot \begin{bmatrix} \bar{A} & \bar{B} \\ I & 0 \end{bmatrix} + \begin{bmatrix} 0 & \bar{C}^T \Pi_{12} \\ * & [\bar{D}^T \Pi_{12}]_s + \Pi_{22} \end{bmatrix} \quad (9)$$

$\varpi_c = (\varpi_1 + \varpi_2)/2$, and Π_{12}, Π_{22} are the upper right and lower right block matrices of Π.

Lemma 2: (Projection Lemma [16]): Let Γ, Λ, Θ be given. There exists a matrix F satisfying $\Gamma F \Lambda + (\Gamma F \Lambda)^T + \Theta < 0$ if and only if the two conditions hold: $\Gamma^\perp \Theta \Gamma^{\perp T} < 0$, $\Lambda^{T^\perp} \Theta \Lambda^{T^{\perp T}} < 0$.

Lemma 3: (Reciprocal Projection Lemma [16]): Let P be any given positive definite matrix. The inequality $\Psi + S + S^T < 0$ is equivalent to the LMI problem

$$\begin{bmatrix} \Psi + P - [W]_s & S^T + W^T \\ * & -P \end{bmatrix} < 0. \quad (10)$$

In this brief, it is assumed that all the state variables can be measured, and we are interested in designing a state feedback controller

$$u(t) = Kx(t) \quad (11)$$

where K is the state feedback gain matrix to be designed. By combining (11) with (4), the closed-loop system is given by

$$\begin{aligned} \dot{x}(t) &= \bar{A}x(t) + \bar{B}w(t) \\ z_1(t) &= \bar{C}x(t) + \bar{D}w(t), \\ z_2(t) &= C_2 x(t) \end{aligned} \quad (12)$$

where

$$\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] = \left[\begin{array}{c|c} A + BK & B_1 \\ \hline C_1 + D_1 K & 0 \end{array}\right]. \quad (13)$$

For the active suspension systems, in accordance with the requirements, the constrained H_∞ control problem is formulated to minimize the H_∞ norm from the disturbance inputs $w(t)$ to the controlled output $z_1(t)$ under the time-domain constraints (6) over the fixed frequency band $\varpi_1 \leq \omega \leq \varpi_2$. By using Lemma 1, we have the following theorem.

Theorem 1: Let positive scalars γ, η, and ρ be given. A state feedback controller in the form of (11) exists, such that the closed-loop system in (12) is asymptotically stable with $w(t) = 0$, and satisfies $\|G(j\omega)\|_\infty^{\varpi_1<\omega<\varpi_2} < \gamma$ for all nonzero $w \in L_2[0, \infty)$, while the constraints in (6) are guaranteed with the disturbance energy under the bound $w_{\max} = (\rho - V(0))/\eta$, if there exist symmetric matrices P, $P_1 > 0$, $Q > 0$ and general matrix F satisfying

$$\begin{bmatrix} -[F]_s & F^T\bar{A} + P_1 & F^T & F^T\bar{B} \\ * & -P_1 & 0 & 0 \\ * & * & -P_1 & 0 \\ * & * & * & -\eta I \end{bmatrix} < 0 \quad (14)$$

$$\begin{bmatrix} -Q & P + j\varpi_c Q - F & 0 & 0 \\ * & -\varpi_1\varpi_2 Q + [F^T\bar{A}]_s & F^T\bar{B} & \bar{C}^T \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0 \quad (15)$$

$$\begin{bmatrix} -I & \sqrt{\rho}K \\ * & -u_{\max}^2 P_1 \end{bmatrix} \leq 0 \quad (16)$$

$$\begin{bmatrix} -I & \sqrt{\rho}\{C_2\}_i \\ * & -P_1 \end{bmatrix} < 0,\ i = 1, 2 \quad (17)$$

where $\varpi_c = (\varpi_1 + \varpi_2)/2$ is a given scalar.

Proof: By using Schur complement, inequality (14) is equivalent to

$$\begin{bmatrix} \frac{1}{\eta}F^T\bar{B}\bar{B}^TF + F^TP_1^{-1}F - [F]_s & F^T\bar{A} + P_1 \\ * & -P_1 \end{bmatrix} < 0. \quad (18)$$

Performing the congruence transformation to inequality (18) by $\mathrm{diag}\{F^{-1}, P_1^{-1}\}$, with $F := W^{-1}$, inequality (18) can be transformed to the following inequality:

$$\begin{bmatrix} \frac{1}{\eta}\bar{B}\bar{B}^T + P_1^{-1} - [W]_s & \bar{A}P_1^{-1} + W^T \\ * & -P_1^{-1} \end{bmatrix} < 0. \quad (19)$$

By using Lemma 3, inequality (19) is equivalent to $\bar{A}P_1^{-1} + P_1^{-1}\bar{A}^T + (1/\eta)\bar{B}\bar{B}^T < 0$, with $\Psi = (1/\eta)\bar{B}\bar{B}^T$ and $S^T = \bar{A}P_1^{-1}$. Clearly, we have

$$\bar{A}^TP_1 + P_1\bar{A} + \frac{1}{\eta}P_1\bar{B}\bar{B}^TP_1 < 0 \quad (20)$$

which can guarantee $\bar{A}^TP_1 + P_1\bar{A} < 0$. From the standard Lyapunov theory for continuous-time linear system, the closed-loop system (12) is asymptotically stable with $w(t) = 0$.

Rewrite inequality (15) as

$$J\Xi J^T + H\Pi H^T + [\Gamma F \Lambda]_s < 0 \quad (21)$$

where

$$\begin{aligned} J &= \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}^T \\ \Xi &= \begin{bmatrix} -Q & P + j\varpi_c Q \\ * & -\varpi_1\varpi_2 Q \end{bmatrix} \\ \Pi &= \begin{bmatrix} I & 0 \\ * & -\gamma^2 I \end{bmatrix} \quad (22) \\ H &= \begin{bmatrix} 0 & \bar{C} & 0 \\ 0 & 0 & I \end{bmatrix}^T \\ \Gamma &= [-I \quad \bar{A} \quad \bar{B}]^T \\ \Lambda &= [0 \quad I \quad 0]. \quad (23) \end{aligned}$$

Then, according to Lemma 2, inequality (21) holds if and only if

$$\begin{aligned} W^T(J\Xi J^T + H\Pi H^T)W &< 0 \\ U(J\Xi J^T + H\Pi H^T)U^T &< 0 \end{aligned} \quad (24)$$

where

$$W = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}^T, \quad U = \begin{bmatrix} \bar{A}^T & I & 0 \\ \bar{B}^T & 0 & I \end{bmatrix}.$$

Note that inequality (24) can be transformed to the following form:

$$\begin{bmatrix} \bar{A} & \bar{B} \\ I & 0 \end{bmatrix}^T \Xi \begin{bmatrix} \bar{A} & \bar{B} \\ I & 0 \end{bmatrix} + \begin{bmatrix} \bar{C} & 0 \\ 0 & I \end{bmatrix}^T \Pi \begin{bmatrix} \bar{C} & 0 \\ 0 & I \end{bmatrix} < 0 \quad (25)$$

which can be further transformed to

$$L + [\bar{C} \quad 0]^T [\bar{C} \quad 0] < 0 \quad (26)$$

where

$$L = \begin{bmatrix} \bar{A} & \bar{B} \\ I & 0 \end{bmatrix}^T \Xi \begin{bmatrix} \bar{A} & \bar{B} \\ I & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\gamma^2 I \end{bmatrix}.$$

By using Schur complement and Lemma 1, we can obtain

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^T \Pi \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \ \varpi_1 \le \omega \le \varpi_2 \quad (27)$$

which is exactly the finite frequency H_∞ performance index inequality in (5).

Denoting $V(t) = x^T(t)P_1x(t)$ as the energy function, and noting that

$$2x^T(t)P_1\bar{B}w(t) \le \frac{1}{\eta}x(t)^TP_1\bar{B}\bar{B}^TP_1x(t) + \eta w(t)^Tw(t) \quad \forall \eta > 0$$

we have

$$\dot{V}(t) \le x(t)^T\left(\bar{A}^TP_1 + P_1\bar{A} + \frac{1}{\eta}P_1\bar{B}\bar{B}^TP_1\right)x(t) + \eta w(t)^Tw(t). \quad (28)$$

According to the inequality in (20), inequality (28) guarantees $\dot{V}(t) \le \eta w(t)^Tw(t)$. Integrating both sides of the above inequality ($\dot{V}(t) \le \eta w(t)^Tw(t)$) from 0 to t results in

$$V(t) - V(0) \le \eta \int_0^t w^T(t)w(t)dt \le \eta \|w\|_2^2 = \eta w_{\max}.$$

This shows that

$$x^T(t)P_1x(t) \le V(0) + \eta w_{\max} = \rho. \quad (29)$$

Consider

$$\begin{aligned} \max_{t\ge 0} |u(t)|^2 &= \max_{t\ge 0} \|Kx(t)\|_2^2 = \max_{t\ge 0} \left\|x^T(t)K^TKx(t)\right\|_2 \\ \max_{t\ge 0} |\{z_2(t)\}_i|^2 &= \max_{t\ge 0} \left\|x^T(t)\{C_2\}_i^T\{C_2\}_i x(t)\right\|_2, \ i = 1, 2. \end{aligned}$$

Using the transformation $\bar{x}(t) = P_1^{1/2}x(t)$, from inequality (29) it follows that $\bar{x}^T(t)\bar{x}(t) \le \rho$. Hence

$$\begin{aligned} \max_{t\ge 0} |u(t)|^2 &= \max_{t\ge 0} \left\|\bar{x}^T(t)P_1^{-1/2}K^TKP_1^{-1/2}\bar{x}(t)\right\|_2 \\ &\le \rho \cdot \lambda_{\max}(P_1^{-1/2}K^TKP_1^{-1/2}), \\ \max_{t\ge 0} |\{z_2(t)\}_i|^2 &\le \rho \cdot \lambda_{\max}(P_1^{-1/2}\{C_2\}_i^T\{C_2\}_i P_1^{-1/2}), \\ & i = 1, 2 \end{aligned} \quad (30)$$

where $\lambda_{\max}(\cdot)$ represents the maximum eigenvalue. Then, the constraints in (6) hold if

$$\begin{aligned} \rho P_1^{-1/2}K^TKP_1^{-1/2} &< u_{\max}^2 I, \\ \rho P_1^{-1/2}\{C_2\}_i^T\{C_2\}_i P_1^{-1/2} &< I, \quad i = 1, 2 \end{aligned} \quad (31)$$

which, by Schur complement, are equivalent to (16) and (17). The proof is completed. □

Since expressions like (14) and (15) involve the forms of FBK, the resulting feasibility problem is nonlinear. Hence, it cannot be handled directly by LMI optimization. In order to solve the nonlinear problem, define

$J_1 = \mathrm{diag}\{F^{-1}, F^{-1}, F^{-1}, I\}$, $J_2 = \mathrm{diag}\{F^{-1}, F^{-1}, I, I\}$, $J_3 = \mathrm{diag}\{I, F^{-1}\}$. Then, we perform a congruence transformation to (14)–(17), respectively, by the full rank matrix J_1^T, J_2^T, J_3^T, and J_3^T on the left, and J_1, J_2, J_3, and J_3 on the right. Defining

$$\bar{Q} = (F^{-1})^T Q F^{-1},\ \bar{P} = (F^{-1})^T P F^{-1}$$
$$\bar{P}_1 = (F^{-1})^T P_1 F^{-1},\ \bar{K} = K F^{-1},\ \bar{F} = F^{-1}$$

the following theorem is obtained.

Theorem 2: Let positive scalars γ, η, and ρ be given. A state feedback controller in the form of (11) exists, such that the closed-loop system in (12) is asymptotically stable with $w(t) = 0$, and satisfies $\|G(j\omega)\|_\infty^{\varpi_1<\omega<\varpi_2} < \gamma$ for all nonzero $w \in L_2[0,\infty)$, while the constraints in (6) are guaranteed with the disturbance energy under the bound $w_{\max} = (\rho - V(0))/\eta$, if there exist matrices $\bar{P}$, $\bar{P}_1 > 0$, $\bar{Q} > 0$, and general matrix $\bar{F}$ satisfying (32)–(35), shown at the bottom of the page. Moreover, if inequalities (32)–(35) have a set of feasible solutions, the control gain K in (11) is given by

$$K = \bar{K}\bar{F}^{-1}.$$

Remark 1: Note that the linear matrix inequality (33) has complex variables. According to [18], the LMI in complex variables can be converted to an LMI of larger dimension in real variables. This means that inequality $S_1 + jS_2 < 0$ is equivalent to $\begin{bmatrix} S_1 & S_2 \\ -S_2 & S_1 \end{bmatrix} < 0$, which implies the LMI in (33) can be addressed.

IV. A Design Example

In this section, we will apply the above approach to design a finite frequency state feedback H_∞ controller based on the quarter-car model described in Section II. The quarter-car model parameters are listed in Table I.

For subsequent comparison, a state feedback H_∞ controller in the finite frequency domain for system (4) is designed first, based on the assumption that all the state variables can be measured. Under zero initial conditions, solve the matrix inequalities (32)–(35) for matrices $\bar{P}$, $\bar{P}_1 > 0$, and $\bar{Q} > 0$ with the optimized parameter $\gamma > 0$ and $\varpi_1 = 4$ Hz, $\varpi_2 = 8$ Hz, $\rho = 0.9$, $\eta = 10\,000$, $z_{\max} = 100$ mm, $u_{\max} = 2500$ N. In the case of optimal γ, an admissible control gain matrix is given based on $K_F = \bar{K}\bar{F}^{-1}$

TABLE I
Quarter-Car Model Parameters

m_s	m_u	k_s	k_t	c_s	c_t
320kg	40kg	18kN/m	200kN/m	1kNs/m	10Ns/m

$$K_F = 10^4 \times [\,0.5033 \quad -1.3155 \quad -0.5329 \quad -0.0547\,].$$

For description in brevity, we denote this finite frequency controller as Controller I hereafter.

Then, we give another H_∞ state feedback controller which is designed over the entire frequency range, that is

$$K_E = 10^4 \times [\,1.3900 \quad 0.4263 \quad -0.0932 \quad -0.0400\,]$$

and we denote this controller as Controller II for brevity.

After obtaining the finite and entire frequency controller, we will compare the two controllers to illustrate the performance of the closed-loop suspension system in finite frequency domain. By the simulation, the responses of the open-loop system, the closed-loop system which is composed of the Controller I and the closed-loop system which is composed of the Controller II, are compared in Fig. 2. In Fig. 2, the solid and dotted lines are the responses of the closed-loop system with finite frequency controller and entire frequency controller, respectively, and the dashed line is the response of the passive system. From the figure, we can see that the finite frequency controller yields the least value of H_∞ norm over the frequency range 4–8 Hz, compared with the passive system and the closed-loop system with an entire frequency controller, which clearly shows that an improved ride comfort has been achieved.

In order to evaluate the suspension characteristics with respect to three performance requirements, we give the disturbance signal as follows to clarify the effectiveness of our finite frequency controller.

$$\begin{bmatrix} -[\bar{F}]_s & A\bar{F} + B\bar{K} + \bar{P}_1 & \bar{F} & B_1 \\ * & -\bar{P}_1 & 0 & 0 \\ * & * & -\bar{P}_1 & 0 \\ * & * & * & -\eta I \end{bmatrix} < 0 \tag{32}$$

$$\begin{bmatrix} -\bar{Q} & \bar{P} + j\varpi_c\bar{Q} - \bar{F} & 0 & 0 \\ * & -\varpi_1\varpi_2\bar{Q} + [A\bar{F} + B\bar{K}]_s & B_1 & \bar{F}^T C_1^T + \bar{K}^T D_1^T \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{33}$$

$$\begin{bmatrix} -I & \sqrt{\rho}\bar{K} \\ * & -u_{\max}^2\bar{P}_1 \end{bmatrix} < 0 \tag{34}$$

$$\begin{bmatrix} -I & \sqrt{\rho}\{C_2\}_i\bar{F} \\ * & -\bar{P}_1 \end{bmatrix} < 0,\ i = 1, 2 \tag{35}$$

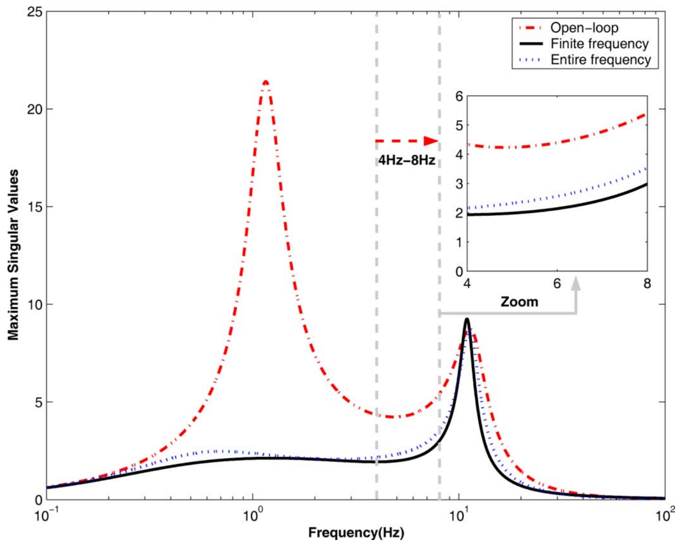

Fig. 2. Frequency response of body vertical acceleration.

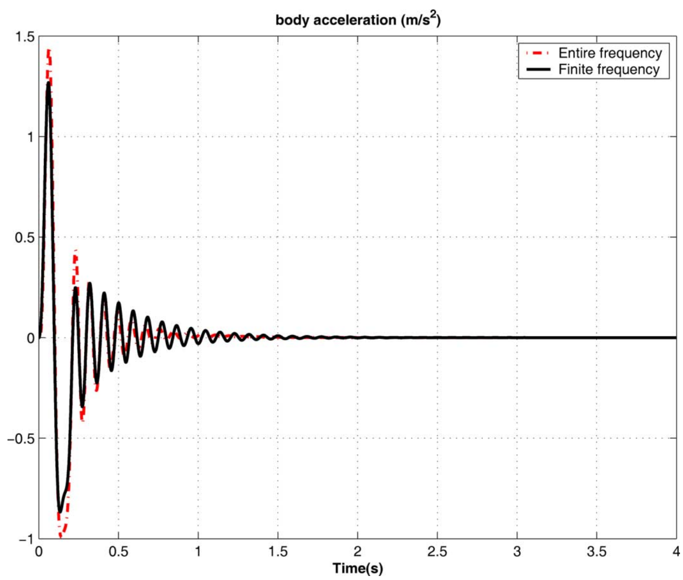

Fig. 3. Time-domain response of body vertical acceleration for active suspension system.

Consider the case of an isolated bump in an otherwise smooth road surface, the disturbance inputs are given by

$$w(t) = \begin{cases} A\sin(2\pi f t), & \text{if } 0 \le t \le T \\ 0, & \text{if } t > T \end{cases} \tag{36}$$

where A, f, and T represent the amplitude, frequency and period of vibration, respectively. Assume $A = 0.5$ m, $f = 5$ Hz (among the frequency band 4–8 Hz) and $T = 1/f = 0.2$ s. The time-domain response of body vertical acceleration for the active suspension system is shown in Fig. 3, where the black solid line and the red dashed line are the responses of body vertical acceleration with the finite frequency controller and the entire frequency controller, respectively. We can clearly see that the value of the body acceleration with the finite frequency controller is less than that with the entire frequency controller. In addition, Fig. 4 shows that the ratio $x_1(t)/z_{\max}$ and the relation dynamic tire load $k_t x_2(t)/(m_s + m_u)g$ are below 1, and the force of the actuator is below the maximum bound $u_{\max}$, which means the time-domain constraints are guaranteed by the designed controller.

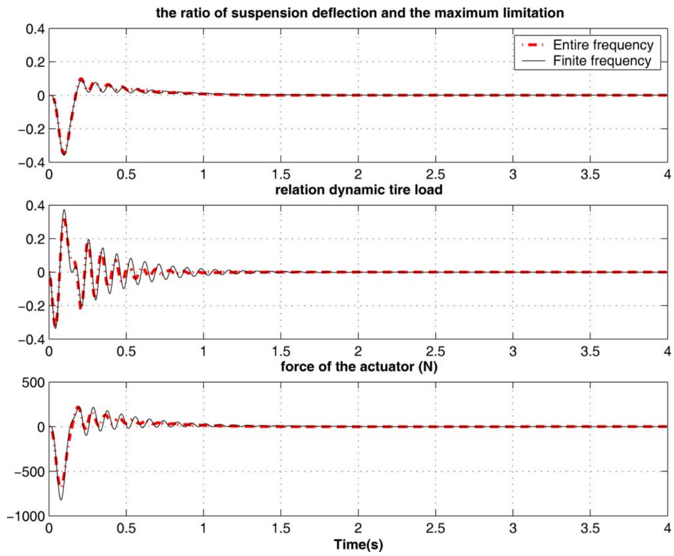

Fig. 4. Time-domain response of constraints for active suspension system.

From Fig. 4, we note that larger actuator forces are needed in the finite frequency control compared with that in the entire frequency control for the reason that the finite frequency control requires more force to match the finite frequency features. However, maybe it is worthwhile to conduct this in exchange for the advantages of finite frequency control. Actuator power consumption is another important issue in automotive active control. In this brief, we can calculate the actuator output average power by the formulation

$$P = \frac{1}{T}\int_0^T (u(t)\cdot s(t))dt$$

where "$s(t)$" respects the displacement of actuator, and according to the installed location, the displacement of actuator is equivalent to that of suspension, that is $s(t) = x_1(t)$, and T is the integral time. In order to show the comparison of the power consumptions between the finite and entire frequency methods, the ratio of the two kinds of powers is calculated, that is

$$\delta = \frac{P_f}{P_e} = \frac{\frac{1}{T_f}\int_0^{T_f}(u_f(t)\cdot s_f(t))dt}{\frac{1}{T_e}\int_0^{T_e}(u_e(t)\cdot s_e(t))dt} = 0.7859$$

where P_f and P_e respect the power consumptions of finite frequency control and entire frequency control, respectively. The above calculation implies the power consumption of actuator in finite frequency control is smaller than the counterpart (power consumption in entire frequency control), which further increases the feasibility of proposed method.

In the literature [14], the H_∞ control of active suspension systems is also considered over the finite frequency domain, and the method used to deal with the problem of finite frequency is to add some weighting functions to the active suspension systems and then the design parameters are chosen such that the weighted system norm is small. This weighting method is effective. However, this method is based on the appropriate weighting function as a precondition, and the choice of weighting function is quite time-consuming, especially when the designer has to shoot for a good tradeoff between the complexity of the weights and the accuracy in capturing desired specifications. In our brief, we provide a more reliable and convenient method to deal with the problem in the finite frequency domain, and avoid using the weighting function. Our simulation results validate that the ride comfort of the closed-loop system composed of the finite frequency controller has been greatly improved, and meanwhile the performance constraints are guaranteed within their allowable bounds.

V. Concluding Remarks

This brief has investigated the problem of H_∞ control with time domain constraints for active vehicle suspension systems in finite frequency domain. By the Generalized KYP lemma, the ride comfort has been improved by minimizing the H_∞ norm in specific frequency band, while the time-domain constraints have also been guaranteed in the framework of linear matrix inequality optimization. Analysis and simulation results for a quarter-car model has shown the effectiveness of the proposed approach.

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